LOCALIZATION OF TWO RETAILERS IN A SINGLE URBAN AREA

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Abstract

The article presents a simple model using both graph theory and the concept of subgame perfect Nash equilibrium in a sequential game of two players from game theory to identify the equilibrium strategies for locating two retail units in one urban area. A brief example of model application is presented for a fictive urban area with seven households forming a path and a cycle.

***Keywords*:** *localization, retailers, Nash equilibrium, game theory*

JEL: *C73, D0, O18*

1 INTRODUCTION

The question of retail unit localization is a crucial one especially for small retailers. This article attempts to formulate a simple model for identifying the equilibrium locations for retail units in the case of urban areas capable of supporting two retail units. In these cases the first retailer to open a retail unit in such an urban area has to take into account all the possible locations available to the second retailer. The problem thus resembles a game of chess, with the first player having to anticipate the response of his opponent and having to act accordingly. One approach capable of modeling such a problem is the concept of a sequential game used in game theory. However, for this approach to become viable, we first have to define the payoff functions of both players. In order to do this, we will use a graph representation of the urban area. Under the assumption that individual consumers will (assuming equal prices) shop at the retail unit located closest to them, we can divide the graph of the urban area in to subgraphs representing the area serviced by each of the two retailers. Further model assumptions are presented in the following section. The problem of identifying the equilibrium location is presented in the third section and a simple application of the model is presented in the fourth section.

2 LOCALIZATION MODEL

First, let us specify the assumptions under which this model will be formulated.

1. A total of N households are located in the urban area.
2. Each household represents a consumer and shops at the closest retailer. If two retailers are located at the same distance from the household, it will shop at both retailers alternately.
3. Each household can become a retailer. When becoming a retailer, the household also remains a consumer as well. Only one retailer can be located at each household.
4. Distances between neighboring households are equidistant and equal to 1.
5. Two retailers A and B will locate their retail units in the urban area.

The urban area can be represented by a graph $G = (V, H)$, where the vertex set V represents individual households and the set of edges H represents the neighboring households. We can divide graph G in to two subgraphs $G_A(V^A, H^A) \subseteq G\ (V, H)$ and $G_B(V^B, H^B) \subseteq G\ (V, H)$. Consumers shopping at retailer A represent the vertex subset $V^A \subseteq V$ and consumers shoping at retailer B the vertex subset $V^A \subseteq V$. Then, the subgraph G_A is an induced subgraph of G by V^A and the subgraph G_B is an induced subgraph of G by V^B.

Using game theory, we can formulate the two retailer localization problem as the following two player sequential game:

$$\left\{\{\mathrm{A}, \mathrm{B}\}; \mathcal{X}^A, \mathcal{X}^B; \Pi^\mathrm{A}, \Pi^\mathrm{B}\right\}, \tag{1}$$

where $\{\mathrm{A}, \mathrm{B}\}$ is the set of players, $\mathcal{X}^A, \mathcal{X}^B$ are the sets of possible strategies of the two players and $\Pi^\mathrm{A}, \Pi^\mathrm{B}$ are the payoff functions of the two players. The set of possible strategies of a player represents the ability to choose any of the N households as the location for his retail unit. We define the set of possible strategies of player A as $\mathcal{X}_\mathrm{A} = \{1, 2, \ldots, N\}$ and of players B as $\mathcal{X}_\mathrm{B} = \{1, 2, \ldots, N\} \setminus \{v_a\}$, with the strategy selected by player A denoted as v_a and the strategy selected by player B denoted v_b. Note that the strategy selected by player A is not included in the set of possible strategies of player B as there can be only one retailer per household, and player A is moving before player B in this sequential game.

The set of consumers shopping at retailer A is defined by the function

$$V^A(G_A) = V^A_{(v_a, v_b)} = \{v_i \in V / d(v_i;\ v_a) \leq d\ (v_i;\ v_b)\}, \tag{2}$$

and the set of consumers shopping at retailer B is defined by the function

$$V^B(G_B) = V^\mathrm{B}_{(v_a, v_b)} = \{v_i \in V / d(v_i;\ v_\mathrm{b}) \leq d\ (v_i;\ v_\mathrm{a})\}. \tag{3}$$

where $d\ (.\,;.)$ is the distance function between two vertices.

The payoff function of a player is then represented by the number of consumers shopping at his retail unit:

$$\Pi^A(G_A) = \Pi^A_{(v_a,v_b)} = \left|V^A{}_{(v_a,\,v_b)}\right| + \frac{\left|V^A_{(v_a,\,v_b)}\right| + \left|V^B_{(v_a,\,v_b)}\right| - |V|}{2}, \qquad (41)$$

$$\Pi^B(G_B) = \Pi^B_{(v_a,\,v_b)} = \left|V^B{}_{(v_a,\,v_b)}\right| + \frac{\left|V^A_{(v_a,\,v_b)}\right| + \left|V^B_{(v_a,\,v_b)}\right| - |V|}{2}. \qquad (52)$$

3 THE EQUILIBRIUM LOCALIZATION PROBLEM

If we assume that both retailers will attempt to maximize their payoff function when selecting the location for their retail unit, the equilibrium localization problem can be solved by finding the subgame perfect Nash equilibrium of the game described in section 2. To find the subgame perfect Nash equilibrium strategies (v_a^*, v_b^*) we first have to find the equilibrium strategy $v^*_{b_{v_a}}$ of player B for every possible strategy $\forall\, v_a \in \mathcal{X}^A$ of player A for which

$$(\forall\, v_b \in \mathcal{X}^B)\left(\Pi^B_{\left(v_a,v^*_{b_{v_a}}\right)} \geq \Pi^B_{(v_a,v_b)}\right) \qquad (6) \qquad 3$$

holds. Using backwards induction player A then selects the strategy v_a^* for which

$$(\forall\, v_a \in \mathcal{X}^A)\left(\Pi^A_{\left(v_a^*,v^*_{b_{v_a}}\right)} \geq \Pi^A_{\left(v_a,v^*_{b_{v_a}}\right)}\right) \qquad (7)$$

holds. Finding the equilibrium strategies (v_a^*, v_b^*) can be also formulated as finding the equilibrium strategy pair (v_a^*, v_b^*) for which

$$\Pi^A_{(v_a^*,v_b^*)} = \max_{v_a \in \mathcal{X}^A}\left(\min_{v_b \in \mathcal{X}^B} \Pi^A_{(v_a,v_b)}\right) \wedge \left(\min_{v_b \in \mathcal{X}^B} \Pi^A_{(v_a^*,v_b)}\right) = \max_{v_a \in \mathcal{X}^A}\left(\min_{v_b \in \mathcal{X}^B} \Pi^A_{(v_a,v_b)}\right) \qquad (8)$$

or

$$\Pi^B_{(v_a^*,v_b^*)} = \min_{v_a \in \mathcal{X}^A}\left(\max_{v_b \in \mathcal{X}^B} \Pi^B_{(v_a,v_b)}\right) \wedge \left(\max_{v_b \in \mathcal{X}^B} \Pi^B_{(v_a^*,v_b)}\right) = \min_{v_a \in \mathcal{X}^A}\left(\max_{v_b \in \mathcal{X}^B} \Pi^B_{(v_a,v_b)}\right) \qquad (9)$$

holds. Note that equations 8 and 9 are equivalent.

4 EXAMPLES OF MODEL APLICATION

As an example let us take a fictive urban area with $N = 7$ households where two competing retailers would like to set up their retail units. The first example will feature these vertices arranged as a path, the second as a cycle.

4.1 Path

As shown in Figure 1, the seven households are forming a path. Table 1 shows all possible payoffs for player A forming a payoff matrix Π^A.

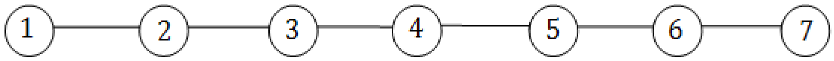

Figure 1: Urban area representing a path

Table 1: Possible payoffs of player *A*

$v_a \backslash v_b$	1	2	3	4	5	6	7
1	N/A	1	1,5	2	2,5	3	3,5
2	6	N/A	2	2,5	3	3,5	4
3	5,5	5	N/A	3	3,5	4	4,5
4	5	4,5	4	N/A	4	4,5	5
5	4,5	4	3,5	3	N/A	5	5,5
6	4	3,5	3	2,5	2	N/A	6
7	3,5	3	2,5	2	1,5	1	N/A

Note that the diagonal of the payoff matrix is not defined since it is not possible for both retail units to be in the same vertex. Solving equation 8 for this payoff matrix we find that this situation has two subgame perfect Nash equilibrium strategy pairs $(v_a^*, v_b^*) = (4; 3)$ and $(v_a^*, v_b^*) = (4; 5)$. Solving equation 9 for the payoff matrix Π^B would lead to the same conclusion as the payoff matrices for both players are complementary.

4.2 Cycle

As shown in Figure 2, the seven households are forming a cycle. Table 2 shows all possible payoffs for player A forming a payoff matrix Π^A.

In case of a cycle, all possible strategy pairs satisfy the conditions of being the subgame perfect Nash equilibrium strategies, and as such, both players are indifferent towards the strategy chosen by the second player.

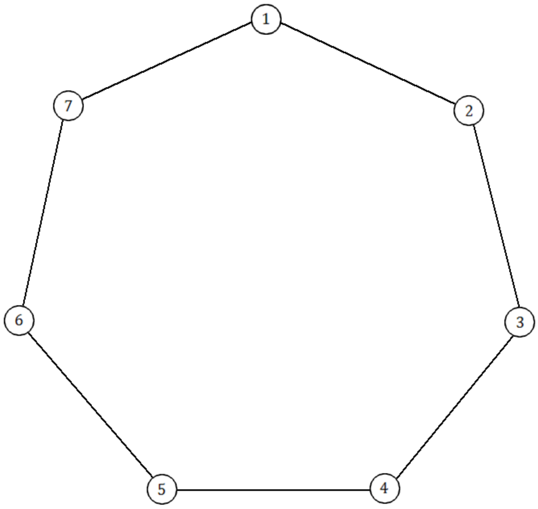

Figure 2: Urban area representing a cycle

Table 2: Possible payoffs of player *A*

$v_a \backslash v_b$	1	2	3	4	5	6	7
1	N/A	3,5	3,5	3,5	3,5	3,5	3,5
2	3,5	N/A	3,5	3,5	3,5	3,5	3,5
3	3,5	3,5	N/A	3,5	3,5	3,5	3,5
4	3,5	3,5	3,5	N/A	3,5	3,5	3,5
5	3,5	3,5	3,5	3,5	N/A	3,5	3,5
6	3,5	3,5	3,5	3,5	3,5	N/A	3,5
7	3,5	3,5	3,5	3,5	3,5	3,5	N/A

5 CONCLUSION

The article presents a model for identifying the optimal retail unit localization in case of two retailers opening a retail unit in a single urban area. As the model utilizes the sequential game form, the concept of subgame perfect Nash equilibrium is used as it accounts for all possible reactions of the player moving as second, and enabling the first player to predict these reactions using backwards induction. An example was provided in the form of a small fictive urban area forming a path and a cycle.

REFERENCES

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